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THE ASYMPTOTIC THEORY OF THERMOELASTIC BEAMS

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The investigation of problems in thin regions leads to lower-dimensional problems: in regions with a small thickness, to plate theories [1–3]; and in small-diameter regions, to beam theories [4–6]. An analogue of the averaging method [7–8] is applicable in this situation. The peculiarities of the problems are clarified in studying the cell problem (which is insufficiently studied in terms of the purely mathematical theory). An analysis of the problem indicates the significant influence of the microstucture on the macroscopic properties of beams and establishes their interrelation. An asymptotic analysis of the thermoelastic problem in a small-diameter region is carried out in this paper. The thermoelastic problem was studied earlier for monolithic composites [9–14] and for for plates [15–17].

Statement of the Problem. Let a linear thermoelastic material occupy a region $\Omega_\varepsilon \subset R^3$ obtained by periodic repetition of a periodicity cell (PC) P_ε along the Ox_1 axis (see Fig. 1). The PC P_ε is obtained by homothetic compression with a coefficient ε of some ε-independent cell P_1; as a result, the geometric dimensions of P_ε remain unchanged as ε is varied. The cell P_1 has congruent lateral faces, parallel to the plane Oy_2y_3, and a piecewise smooth free boundary γ (the last property is immediately extended to the boundary P_ε of the PC and the boundary of Ω_ε). Otherwise, the cell P_ε is rather arbitrary; in particular, it can be multi-connected (for example, in farms).

The equilibrium equations for this three-dimensional body with fixed lateral faces (see Fig. 1) have the form [18]

$$\int\limits_{\Omega_\varepsilon} \sigma^\varepsilon_{ij} v_{i,j}\, d\mathbf{x} + \int\limits_{\Gamma_\varepsilon} (\varepsilon^{-2}\mathbf{g})\mathbf{v}\, d\mathbf{x} = \int\limits_{\Omega_\varepsilon} (\varepsilon^{-2}\mathbf{f})\mathbf{v}\, d\mathbf{x} \tag{1}$$

for any $\mathbf{v} \in V(\Omega_\varepsilon) = \{\mathbf{v} \in H^1(\Omega_\varepsilon) : \mathbf{v}(\mathbf{x}) = 0 \text{ at } x_1 = \pm 1\}$. For the definition of H^1 see, for example, [19]. Here σ^ε_{ij} are the local stresses related to the local displacements $\mathbf{u}^\varepsilon$ by the Duhamel-Neumann law [18]:

$$\sigma^\varepsilon_{ij} = \varepsilon^{-4} a_{ijkl}(\mathbf{x}/\varepsilon) u^\varepsilon_{k,l} + \varepsilon^{-4}\beta^{(-4)}(\mathbf{x}/\varepsilon)\theta, \tag{2}$$

where θ is temperature (we assume it to be a smooth known function); and $\{a_{ijkl}\}$ is the tensor of elastic constants; $\{\beta_{ij}\}$ is the tensor of thermoelastic constants; and $\beta_{ij} = \varepsilon^{-4} a_{ijkl}\alpha_{kl}$ ($\{\alpha_{kl}\}$ are the coefficients of thermal expansion).

Remark 1 (the choice of orders). Orders are chosen under the asumption that in the limit $\varepsilon \to 0$, the beam has nonzero stiffness and the forces are not equal to zero: the volume and surface forces are of order ε^{-2}, and elastic constants are of order ε^{-4}. The problem of orders is of prime importance. Thus, the classical thermoelastic relations have been derived in [15, 16] for plates with thermal expansion coefficients of order ε. This choice of orders excludes the usual axial elongation. In our case, with the choice of β_{ij} of order ε^{-4} (correspondingly, α_{ij} is of the order of one) this elongation is taken into account, but to equilibrate arising moments, the beam deflections have to be of order ε^{-1}. The functions $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$ are considered independent of ε.

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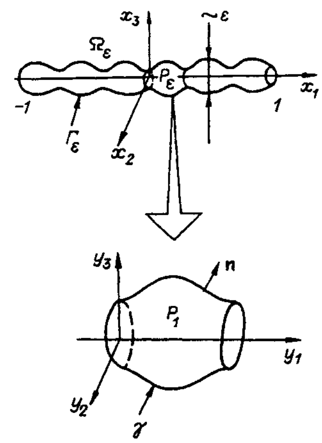

Fig. 1

The periodicity of the beam microstructure is taken into account by the fact that the functions $a_{ijkl}(\mathrm{x}/\varepsilon)$ and $\beta_{ij}(\mathrm{x}/\varepsilon)$ are periodic in x_1 with period εm ($[0, \varepsilon m]$ is the projection of the PC onto the axis Ox_1).

Asymptotic Expansion. The asymptotic expansion has the form:
for the displacements:

$$\mathbf{u}^\varepsilon = \varepsilon^{-1} w_\alpha^{(-1)}(x_1)\mathbf{e}_\alpha - y_\alpha w_{\alpha,1x}^{(-1)}(x_1)\mathbf{e}_1 + \varepsilon^{-1} y_{\tilde\beta} s_\beta \mathbf{e}_\beta \varphi^{(-1)}(x_1) + \mathbf{u}^{(0)}(x_1) + \varepsilon \mathbf{u}^{(1)}(x_1, \mathbf{y}) + \dots \tag{3}$$

$[\tilde\beta = 2$ at $\beta = 3,\ \ \tilde\beta = 3$ at $\beta = 2,\ \ s_1 = 0,\ \ s_2 = 1,\ \ s_3 = -1\ (\beta = 2, 3)]$;
for the test function:

$$\mathbf{v} = \mathbf{v}^{(0)}(x_1) + \varepsilon \mathbf{v}^{(1)}(x_1, \mathbf{y}) + \dots;$$

for the stresses:

$$\sigma_{ij}^\varepsilon = \varepsilon^{-4}\sigma_{ij}^{(-4)}(x_1, \mathbf{y}) + \varepsilon^{-3}\sigma_{ij}^{(-3)}(x_1, \mathbf{y}) + \dots,$$

where $x_1 \in [-1, 1]$ is a slow variable and $\mathbf{y} = \mathbf{x}/\varepsilon$ are fast variables [7, 8]; and $\{\mathbf{e}_i\}$ is the ith basis vector of the standard rectangular system of coordinates. The functions on the right-hand sides of (3) are assumed to be periodic in y_1 with period m. Note that the first three terms in (3) correspond to the local problems of bending and torsion of a beam (see below). Here and below, the Greek indices take on values 2 and 3 and the Latin ones takes values 1, 2, 3.

The operator of differentiation $\partial/\partial x_i$ for functions of the form $f(x_1, \mathbf{y})$ can be written as

$$\frac{\partial}{\partial x_1} + \varepsilon^{-1}\frac{\partial}{\partial y_1}, \qquad \varepsilon^{-1}\frac{\partial}{\partial y_\alpha} \quad (\alpha = 2, 3). \tag{4}$$

After using (4), from (1), we obtain

$$\int_{\Omega_\varepsilon} (\sigma_{ij}^\varepsilon \varepsilon^{-1} v_{i,jy} + \sigma_{i1}^\varepsilon v_{i,1x})\, d\mathbf{x} + \int_{\Gamma_\varepsilon} \varepsilon^{-2}\mathbf{g}\mathbf{v}\, d\mathbf{x} = \int_{\Omega_\varepsilon} \varepsilon^{-2}\mathbf{f}\mathbf{v}\, d\mathbf{x}. \tag{5}$$

Here and below, we use the notation $,jy = \partial/\partial y_j$ and $,1x = \partial/\partial x_1$.

We change the variables in (5):

$$\mathbf{x} \to \mathbf{z} = (x_1, y_2, y_3) = (x_1, x_2/\varepsilon, x_3/\varepsilon).$$

Then

$$\varepsilon^2 \int\limits_{\Omega_1} (\sigma_{ij}^{\varepsilon}\varepsilon^{-1} v_{i,jy} + \sigma_{i1}^{\varepsilon} v_{i,1x})\, dz + \int\limits_{\Gamma_1} \mathbf{g}\mathbf{v}\, dz = \int\limits_{\Omega_1} \mathbf{f}\,\mathbf{v}\, dz, \tag{6}$$

where Ω_1 and Γ_1 are the regions obtained from Ω_ε and Γ_ε by increasing their cross section by a factor of ε^{-1} (the diameters of Ω_1 and Γ_1 are of the order of one). Substitution of (3) into (6) yields

$$\sum_{k=0}^{\infty}\sum_{m=-4}^{\infty} \int\limits_{\Omega_1} \varepsilon^2(\varepsilon^{m+k-1}\sigma_{ij}^{(m)} v_{i,jy}^{(k)} + \varepsilon^{m+k}\sigma_{i1}^{(m)} v_{i,1x}^{(k)})\, dz + \sum_{k=0}^{\infty} \int\limits_{\Gamma_1} \varepsilon^k \mathbf{g}\mathbf{v}^{(k)}\, dz = \sum_{k=0}^{\infty} \int\limits_{\Omega_1} \varepsilon^k \mathbf{f}\,\mathbf{v}^{(k)}\, dz. \tag{7}$$

After substituting (3) into the Duhamel–Neumann law and equating expressions with equal powers of ε, we obtain

$$\sigma_{ij}^{(m)} = a_{ijkl}(\mathbf{x}/\varepsilon) u_{k,ly}^{(m+5)} + a_{ijk1}(\mathbf{x}/\varepsilon) u_{k,1x}^{(m+4)} + \begin{cases} 0, & \\ \beta_{ij}^{(-4)}(\mathbf{x}/\varepsilon)\theta & \text{at } m=-4. \end{cases} \tag{8}$$

Here and below in sums $m = -4, -3, \ldots$, $k = 0, 1, \ldots$.

It is necessary to take into account that

$$\varepsilon^{-4}[a_{ij\alpha 1}\varepsilon^{-1} w_{\alpha,1x}^{(-1)}(x_1) - a_{ij1\alpha}\varepsilon^{-1}(y_\alpha w_\alpha^{(-1)}(x_1))_{,\alpha y}] = \varepsilon^{-5}[a_{ij\alpha 1} - a_{ij1\alpha}] = 0,$$

$$\varepsilon^{-4}[\varepsilon^{-1} a_{ijkl}(s_\beta y_{\bar\beta} \mathbf{e}_\beta \varphi(x_1))_{,ly}] = \varepsilon^{-5}[a_{ij23} - a_{ij32}] = 0$$

by virtue of the symmetry of the elastic constants [18]. It follows from these equalities that terms of order ε^{-5} do not appear in the expansion of stresses (3)

Derivation of Equilibrium Equations. The principal aim of this paper is to get a boundary value problem for functions appearing in the asymptotic expansion (3). To do so, we consider problem (7), (8) for some choice of k, m and a special choice of the test function $\mathbf{v}$.

Remark 2. We introduce the notation

$$\langle f\rangle = \frac{1}{m}\int\limits_{P_1} f(\mathbf{y})\, d\mathbf{y} \text{ — the mean value } P_1 \text{ of the PC in fast variables,}$$

$$\langle f\rangle_\gamma = \frac{1}{m}\int\limits_{\gamma} f(\mathbf{y})\, dy \text{ — the mean value } P_1 \text{ on the lateral surface } \gamma \text{ of the PC .}$$

The following relations hold as $\varepsilon \to 0$:

$$\int\limits_{\Omega_1} f(x_1, \mathbf{y})\, d\mathbf{z} \to \int\limits_{-1}^{1} \langle f\rangle(x_1)\, dx_1, \qquad \int\limits_{\Gamma_1} f(x_1, \mathbf{y})\, d\mathbf{z} \to \int\limits_{-1}^{1} \langle f\rangle_\gamma(x_1)\, dx_1.$$

Let $k = 0$ [i.e., we assume that in (3) $\mathbf{v}^{(k)} = 0$ for $k = 1, 2, \ldots$ which is possible by virtue of the arbitrariness of $\{\mathbf{v}^{(k)}\}$]. For these k and $\mathbf{v}$, it follows from (7) that

$$\sum_{m=-4}^{\infty} \varepsilon^2 \int\limits_{\Omega_1} \varepsilon^m \sigma_{i1}^{(m)} v_{i,1x}^{(0)}\, dz + \int\limits_{\Gamma_1} \mathbf{g}\mathbf{v}^{(0)}\, dz = \int\limits_{\Omega_1} \mathbf{f}\,\mathbf{v}^{(0)}\, dz.$$

We introduce the quantaties $N_{ij}^{(m)} = \langle \sigma_{ij}^{(m)}\rangle$ (with the meaning of axial moments [5]) and $M_{i\alpha}^{(m)} = \langle y_\alpha \sigma_{i1}^{(m)}\rangle$ (which have the meaning of bending moments). In the above notation, using Remark 2 and equating terms with identical nonpositive powers of ε in the last equality, we get

$$N_{i1,1x}^{(-4)} = 0, \quad N_{i1,1x}^{(-3)} = 0, \quad N_{i1,1x}^{(-2)} = \langle g_i\rangle_\gamma + \langle f_i\rangle. \tag{9}$$

We set in (7) $k = 1$ (i.e., all summands in expression (3) for the test function $\mathbf{v}$ are equal to zero, except for $\mathbf{v}^{(1)}$) and choose $\mathbf{v}^{(1)}$ as

$$\mathbf{v}^{(1)}(x_1, y) = y_2\mathbf{v}_2(x_1) + y_3\mathbf{v}_3(x_1). \tag{10}$$

Here $\mathbf{v}_2, \mathbf{v}_3 \in C^1([-1,1])$ and $\mathbf{v}_2(\pm 1) = \mathbf{v}_3(\pm 1) = 0$. With this choice of k and $\mathbf{v}$, Eq. (7) takes the form

$$\sum_{m=-4}^{\infty} \int_{\Omega_1} \{\varepsilon^m \sigma_{ij}^{(m)}(v_{2i}\delta_{2j} + v_{3i}\delta_{3j}) + \varepsilon^{m+1}\sigma_{i1}^{(m)}(y_2 v_{2i,1x} + y_3 v_{3i,1x})\} d\mathbf{z}$$

$$+ \int_{\Gamma_1} \mathbf{g}(y_2\mathbf{v}_2 + y_3\mathbf{v}_3) d\mathbf{z} = \int_{\Omega_1} \mathbf{f}(y_2\mathbf{v}_2 + y_3\mathbf{v}_3) d\mathbf{z}. \tag{11}$$

Making use of Remark 2 and equating expressions with identical nonpositive powers of ε, from (11), we obtain

$$\begin{aligned} &N_{i\alpha}^{(-4)} = 0, \quad -M_{\alpha i,1x}^{(-4)} + N_{i\alpha}^{(-3)} = 0 \qquad \text{for } m = -4; \\ &-M_{\alpha i.1x}^{(-3)} + N_{i\alpha}^{(-2)} = \langle g_i y_\alpha \rangle_\gamma + \langle f_i y_\alpha \rangle \qquad \text{for } m = -3 \end{aligned} \tag{13}$$

($\alpha = 2, 3$). Relations (9), (12), and (13) are the equilibrium eqautions of a one-dimensional beam.

Derivation of Governing Relations. Let us now establish relations between the forces and moments introduced above and the strain characteristics of a beam. Note that the main peculiarities of problems in thin regions become more pronounced in derivation of limiting governing relations [1–6].

We set $k = 1$, $m = -4$ and choose the test function in the form $\mathbf{v} = \varepsilon\mathbf{v}^{(1)}(\mathbf{y})$ [the other $\mathbf{v}^{(k)}$ in (3) are equal to zero]. For $\mathbf{v}$ thus chosen, from (7), we obtain the problem

$$\sigma_{ij,jy}^{(-4)} = 0 \quad \text{in } \Omega_1, \qquad \sigma_{ij}^{(-4)} n_j = 0 \quad \text{on } \Gamma_1, \tag{14}$$

where $\mathbf{n}$ is the normal to the lateral surface of the region Ω_1.

We make use of relation (8) for $m = -4$. Substituting it into (14) with allowance for (3) yields the equation

$$(a_{ijkl}(\mathbf{y})u_{k,ly}^{(1)} + a_{ijp1}(\mathbf{y})u_{p,1x}^{(0)}(x_1) + a_{ij1\alpha}(\mathbf{y})w_{\alpha,1x1x}^{(-1)}$$

$$+ s_\beta a_{ij\beta\bar{\beta}}(\mathbf{y})\varphi_{,1x}^{(-1)}(x_1) + \beta_{ij}^{(-4)}(\mathbf{y})\theta(x_1))_{,jy} = 0 \quad \text{in } \Omega_1 \tag{15}$$

with the boundary condition

$$(a_{ijkl}(\mathbf{y})u_{k,ly}^{(1)} + a_{ijp1}(\mathbf{y})u_{p,1x}^{(0)}(x_1) + a_{ij1\alpha}(\mathbf{y})w_{\alpha,1x1x}^{(-1)}(x_1)$$

$$+ s_\beta a_{ij\beta\bar{\beta}}(\mathbf{y})\varphi_{,1x}^{(-1)}(x_1) + \beta_{ij}^{(-4)}(\mathbf{y})\theta(x_1))n_j = 0 \quad \text{on } \Gamma_1 \tag{16}$$

and the requirement of periodicity: $\mathbf{u}^{(1)}(x_1, \mathbf{y})$ is a periodic function of y_1 of period m. By Γ_1 is denoted the lateral surface of PC Ω_1 (see Fig. 1).

In the problem in variable $\mathbf{y}$, functions of x_1 play the role of parameters [1, 19]. In view of this, problem (15), (16) leads to the cell problems (CPr) of beam theory. Here appear problems corresponding to torsion and thermal deformations, which are new compared to CPr for monolithic composites [7, 8] and plates [1, 2].

We now introduce the function $\mathbf{X}^{1p}(\mathbf{y})$ as a solution of "the first elastic CPr of beam theory"

$$(a_{ijkl}(\mathbf{y})X_{k,ly}^{1p} + a_{ijp1}(\mathbf{y}))_{,jy} = 0 \quad \text{in } P_1, \qquad (a_{ijkl}(\mathbf{y})X_{k,ly}^{1p} + a_{ijp1}(\mathbf{y}))n_j = 0 \quad \text{on } \gamma \tag{17}$$

($\mathbf{X}^{1p}$ is a periodic function of y_1 of period m, $\langle \mathbf{X}^{1p} \rangle = 0$).

We introduce the function $\mathbf{X}^3(\mathbf{y})$ as a solution of the "CPr of beam torsion"

$$(a_{ijkl}(\mathbf{y})X_{k,ly}^{3} + a_{ij\beta 1}(\mathbf{y})s_\beta y_{\bar{\beta}})_{,jy} = 0 \quad \text{in } P_1, \qquad (a_{ijkl}(\mathbf{y})X_{k,ly}^{3} + a_{ij\beta 1}(\mathbf{y})s_\beta y_{\bar{\beta}})n_j = 0 \quad \text{on } \gamma \tag{18}$$

$[\mathbf{X}^3(\mathbf{y})$ is a periodic function of y_1 with period m, $\langle \mathbf{X}^3 \rangle = 0]$.

We introduce the function $\boldsymbol{\mathcal{F}}(\mathbf{y})$ as solution of "the first thermoelastic PC of beam theory":

$$(a_{ijkl}(\mathbf{y})\mathcal{F}_{k,ly} + \beta_{ij}^{(-4)}(\mathbf{y}))_{,jy} = 0 \quad \text{in } P_1, \qquad (a_{ijkl}(\mathbf{y})\mathcal{F}_{k,ly} + \beta_{ij}^{(-4)}(\mathbf{y}))n_j = 0 \quad \text{on } \gamma \tag{19}$$

$[\boldsymbol{\mathcal{F}}(\mathbf{y})$ is a periodic function of y_1 with period m, $\langle \boldsymbol{\mathcal{F}} \rangle = 0]$.

By virtue of the linearity of problem (15), (16) and the remark on functions of x_1, the solution of (15), (16) can be represented as a linear combination of solutions of CPr (17)–(19) in the form

$$\mathbf{u}^{(1)} = \mathbf{X}^{1p}(\mathbf{y})u_{p,1x}^{(0)}(x_1) + \mathbf{X}^{2\alpha}(\mathbf{y})w_{\alpha,1x}^{(-1)}(x_1) + \boldsymbol{\mathcal{F}}(\mathbf{y})\theta(x_1) + \mathbf{X}^3(\mathbf{y})\varphi_{,1x}^{(-1)}(x_1) + \mathbf{U}(x_1). \tag{20}$$

$\mathbf{U}(x_1)$ appear in (20) because (15), (16) contain only derivatives with respect to $\mathbf{y}$.

Some part of solutions of CPr (17) is found in an explicit way, namely [5]:

$$X_k^{1\alpha}(\mathbf{y}) = -\delta_{1k}y_\alpha \quad (\alpha = 2, 3). \tag{21}$$

Note that (20) is a particular solution of the problem (15) and (16). This distinguishes the problem for a beam from the problem for a plate. To derive the general solution of (15) and (16), we consider a homogeneous problem corresponding to (15) and (16).

$$(a_{ijkl}(\mathbf{y})X_{k,ly})_{,jy} = 0 \quad \text{in } P_1, \qquad a_{ijkl}(\mathbf{y})X_{k,ly}n_j = 0 \quad \text{on } \gamma$$

$[\mathbf{X}(\mathbf{y})$ is a periodic function of y_1 with period m, $\langle \mathbf{X} \rangle = 0]$.

The solution of this problem is $\mathbf{X} = y_{\bar{\beta}}s_\beta \mathbf{e}_\beta \varphi(x_1)$ for any function $\varphi(x_1)$. Indeed,

$$a_{ijkl}(y_{\bar{\beta}}s_\beta \mathbf{e}_\beta \varphi(x_1))_{k,ly} = (a_{ij23} - a_{ij32})\varphi(x_1) = 0, \tag{22}$$

since by virtue of the symmetry of elastic constants [18], $a_{ij23} = a_{ij32}$. Note that, in the absence of this symmetry, the situation is radically changed [20–23].

As is seen, the solution of the homogeneous CPr introduces formally one more degree of freedom — the torsion of a beam. For monolithic composites and plates, the solution of the homogeneous problem is equal to zero [1, 7].

Taking into account the previous results, the general solution of the problem (15), (16) is the sum of (20) and of the solution of the homogeneous problem and can be written in the coordinate form

$$\begin{aligned} u_1^{(1)} &= X_1^{11}(\mathbf{y})u_{1,1x}^{(0)}(x_1) - y_\alpha u_{\alpha,1x}^{(0)}(x_1) + X_1^{2\alpha}(\mathbf{y})w_{\alpha,1x}^{(-1)}(x_1) + \mathcal{F}_1(\mathbf{y})\theta(x_1) + X_1^3(\mathbf{y})\varphi_{,1x}^{(-1)}(x_1) + U_1(x_1), \\ u_\beta^{(1)} &= X_\beta^{11}(\mathbf{y})u_{1,1x}^{(0)}(x_1) + \mathcal{F}_\beta(\mathbf{y})\theta(x_1) + X_\beta^{2\alpha}(\mathbf{y})w_{\alpha,1x}^{(-1)}(x_1) + X_\beta^3(\mathbf{y})\varphi_{,1x}^{(-1)}(x_1) + y_{\bar{\beta}}s_\beta\varphi(x_1) + U_\beta(x_1). \end{aligned} \tag{23}$$

Substituting (23) into (8) for the case $m = -4$, after calculations, we get

$$\begin{aligned} \sigma_{ij}^{(-4)} &= a_{ij11}(\mathbf{y})u_{1,1x}^{(0)}(x_1) + a_{ijkl}(\mathbf{y})X_{k,ly}^{11}(y)u_{1,1x}^{(0)}(x_1) + a_{ij\beta\bar{\beta}}(\mathbf{y})s_\beta\varphi(x_1) + a_{ijkl}(\mathbf{y})\mathcal{F}_{k,ly}(y)\theta(x_1) \\ &\quad + \beta_{ij}^{(-4)}(\mathbf{y})\theta(x_1) + a_{ij1\alpha}(\mathbf{y})w_{\alpha,1x1x}^{(-1)}(x_1) + a_{ijkl}(\mathbf{y})X_{k,ly}^{2\alpha}(\mathbf{y})w_{\alpha,1x1x}^{(-1)}(x_1) \\ &\quad + a_{ij\beta\bar{\beta}}(\mathbf{y})y_{\bar{\beta}}s_\beta\varphi_{,1x}^{(-1)}(x_1) + a_{ijkl}(\mathbf{y})X_{k,ly}^3(\mathbf{y})\varphi_{,1x}^{(-1)}(x_1). \end{aligned} \tag{24}$$

Integrating first (24) and then (24) multiplied by y_β (taking into account that in this procedure functions of x_1 behave as parameters), we get, for isotropic materials,

$$\begin{aligned} N_{11}^{(-4)} &= A_{11}^{00}u_{1,1x}^{(0)} + A_{11\alpha}^{01}w_{\alpha,1x1x}^{(-1)} + B_{11}^0\theta + I_{11}\varphi_{,1x}^{(-1)}, \\ M_{i\beta}^{(-4)} &= A_{i\beta}^{10}u_{1,1x}^{(0)} + A_{i\beta\alpha}^{11}w_{\alpha,1x1x}^{(-1)} + B_{i\beta}^1\theta + J_{i\beta}\varphi_{,1x}^{(-1)} \end{aligned} \tag{25}$$

($a_{i1\beta\bar{\beta}} = 0$ for isotropic materials [18]). Here

$$A^{00}_{ij} = \langle a_{ij11}(\mathbf{y}) + a_{ijkl}(\mathbf{y}) X^{11}_{k,ly}(\mathbf{y})\rangle, \qquad A^{01}_{i\beta\alpha} = \langle a_{i\beta 1\alpha}(\mathbf{y}) + a_{i\beta kl}(\mathbf{y}) X^{2\alpha}_{k,ly}(\mathbf{y})\rangle,$$
$$A^{10}_{i\beta} = \langle (a_{i111}(\mathbf{y}) + a_{i\beta kl}(\mathbf{y}) X^{2\alpha}_{k,ly}(\mathbf{y})) y_\beta\rangle, \qquad A^{11}_{i\beta\alpha} = \langle (a_{i11\alpha}(\mathbf{y}) + a_{i1kl}(\mathbf{y}) X^{2\alpha}_{k,ly}(\mathbf{y})) y_\beta\rangle,$$
$$B^{0}_{ij} = \langle \beta^{(-4)}_{ij}(\mathbf{y}) + a_{ijkl}(\mathbf{y}) \mathcal{F}_{k,ly}(\mathbf{y})\rangle, \qquad B^{1}_{i\beta} = \langle (\beta^{(-4)}_{i1}(\mathbf{y}) + a_{i1kl}(\mathbf{y}) \mathcal{F}_{k,ly}(\mathbf{y})) y_\beta\rangle, \tag{26}$$
$$I_{ij} = \langle a_{ijkl}(\mathbf{y}) X^{3}_{k,ly}(\mathbf{y})\rangle, \qquad J_{i\beta} = \langle a_{i1kl}(\mathbf{y}) X^{3}_{k,ly}(\mathbf{y}) y_\beta\rangle.$$

The equilibrium equations for determining $u^{(0)}_1$, $w^{(-1)}_\alpha$, $\varphi^{(-1)}$ are of the form [see (9), (12)]

$$\begin{aligned} &N^{(-4)}_{11,1x} = 0 && \text{(for axial forces)},\\ &M^{(-4)}_{1\alpha.1x1x} = 0 && \text{(for bending moments)},\\ &\mathcal{M}_{,1x} = 0 && \text{(for the twisting moment)} \end{aligned} \tag{27}$$

($\mathcal{M} = M^{(-4)}_{23} - M^{(-4)}_{32}$ is the torsion moment). To obtain the second equation from (24) it is neseccary to differentiate (12) for $i = 1$ and use (9). The third equation of (27) follows from (12) and from the equality $N^{(-3)}_{23} = N^{(-3)}_{32}$, which is a consequence of the symmetry of the stress tensor.

Boundary conditions follow from expansion (2):

$$u^{(0)}_1(\pm 1) = w^{(-1)}_\alpha(\pm 1) = w^{(-1)}_{\alpha,1x}(\pm 1) = \varphi^{(-1)}(\pm 1) = 0. \tag{28}$$

The solution of problem (27), (28) in the general case is not equal to zero, because of the presence of thermoelastic terms.

Derivation of Higher-Order Terms of the Asymptotic Expansion. Setting $k = 1$, $m = -3$ in (7) and taking the test function as $\mathbf{v} = \varepsilon \mathbf{v}^{(1)}(\mathbf{y})$, we get the problem

$$\sigma^{(-3)}_{ij,jy} = 0 \quad \text{in } \Omega_1, \qquad \sigma^{(-3)}_{ij} n_j = 0 \quad \text{on } \Gamma_1. \tag{29}$$

For $m = -3$, Eq. (8) takes the form

$$\sigma^{(-3)}_{ij} = a_{ijkl}(\mathbf{y}) u^{(2)}_{k,ly} + a_{ijk1}(\mathbf{y}) u^{(1)}_{k,1x}. \tag{30}$$

Substituting into (29) expressions (27) for $\mathbf{u}^{(1)}$, we get

$$\sigma^{(-3)}_{ij} = a_{ijkl}(\mathbf{y}) u^{(2)}_{k,ly} + a_{ijk1}(\mathbf{y}) X^{11}_k(\mathbf{y}) u^{(0)}_{1,1x1x}(x_1) - a_{ij11}(\mathbf{y}) y_\alpha u^{(0)}_{\alpha,1x1x}(x_1) + a_{ijk1}(\mathbf{y}) \mathcal{F}_k(\mathbf{y}) \theta_{,1x}(x_1)$$
$$+ a_{ijk1}(\mathbf{y}) U_{k,1x}(x_1) + a_{ij\beta\bar{\beta}}(\mathbf{y}) s_\beta \varphi_{,1x}(x_1) + a_{ijk1}(\mathbf{y}) X^{1\alpha}_k(\mathbf{y}) w^{(-1)}_{\alpha,1x}(x_1) + a_{ijk1}(y) X^3_k(\mathbf{y}) \varphi^{(-1)}_{,1x1x}(x_1). \tag{31}$$

Moreover,

$$\mathbf{u}^{(2)}(x_1, \mathbf{y}) \quad \text{should be a periodic function of } y_1 \text{ with period } m,\ \langle \mathbf{u}^{(2)}\rangle = 0. \tag{32}$$

We introduce the function $\mathbf{Y}^{(p)}(\mathbf{y})$ as a solution of "the second PC of beam theory:"

$$\begin{aligned} &(a_{ijkl}(\mathbf{y}) Y^{(p)}_{k,ly} + a_{ijk1}(\mathbf{y}) X^{1p}_k(\mathbf{y}))_{,jy} = 0 && \text{in} \quad P_1,\\ &(a_{ijkl}(\mathbf{y}) Y^{(p)}_{k,ly} + a_{ijk1}(\mathbf{y}) X^{1p}_k(\mathbf{y})) n_j = 0 && \text{on} \quad \gamma \end{aligned} \tag{33}$$

($\mathbf{Y}^{(p)}$ is a periodic function of y_1 with period m, $\langle \mathbf{Y}^{(p)}\rangle = 0$),
the function $\mathbf{T}(\mathbf{y})$ as the solution of "the second PC of thermoelastic beams":

$$\begin{aligned} &(a_{ijkl}(\mathbf{y}) T_{k,ly} + a_{ijk1}(\mathbf{y}) \mathcal{F}_k(\mathbf{y}))_{,jy} = 0 && \text{in } P_1,\\ &(a_{ijkl}(\mathbf{y}) T_{k,ly} + a_{ijk1}(\mathbf{y}) \mathcal{F}_k(\mathbf{y})) n_j = 0 && \text{on } \gamma \end{aligned} \tag{34}$$

($\mathbf{T}$ is a periodic function of y_1 with period m, $\langle \mathbf{T}\rangle = 0$), the function $\mathbf{Z}(\mathbf{y})$ as a solution of "the second PC of beam torsion":

$$(a_{ijkl}(\mathbf{y})Z_{k,ly} + a_{ijk1}(\mathbf{y})X_k^3(\mathbf{y}))_{,jy} = 0 \qquad \text{in } P_1,$$
$$(a_{ijkl}(\mathbf{y})Z_{k,ly} + a_{ijk1}(\mathbf{y})X_k^3(\mathbf{y}))n_j = 0 \qquad \text{on } \gamma \tag{35}$$

($\mathbf{Z}$ is a periodic function of y_1 with period m, $\langle \mathbf{Z} \rangle = 0$).

With the help of the PC (33)–(35) the solution of (29)–(31) can be written as

$$\mathbf{u}^{(2)} = \mathbf{X}^{1p}(\mathbf{y})U_{p,1x}(x_1) - \mathbf{X}^{2\alpha}(\mathbf{y})u^{(0)}_{\alpha,1x1x}(x_1) + \mathbf{T}(\mathbf{y})\theta_{,1x}(x_1)$$
$$+\mathbf{Y}^{(1)}(\mathbf{y})u^{(0)}_{1,1x1x}(x_1) + \mathbf{X}^3(\mathbf{y})\varphi_{,1x}(x_1) + \mathbf{Y}^{(\alpha)}(\mathbf{y})w^{(-1)}_{\alpha,1x}(x_1) + \mathbf{Z}(\mathbf{y})\varphi^{(-1)}_{,1x1x}(x_1). \tag{36}$$

Substitution of (36) into (31) with allowance for (21) yields

$$\sigma_{ij}^{(-3)} = (a_{ij11}(\mathbf{y}) + a_{ijkl}(\mathbf{y})X^{11}_{k,ly}(\mathbf{y}))U_{1,1x}(x_1) - (a_{ij11}(\mathbf{y})y_\alpha + a_{ijkl}(\mathbf{y})X^{2\alpha}_{k,ly}(\mathbf{y}))u^{(0)}_{\alpha,1x1x}(x_1)$$
$$+ (a_{ijkl}(\mathbf{y})Y^{(1)}_{k,ly}(y) + a_{ijk1}(\mathbf{y})X^{11}_k(y))u^{(0)}_{1,1x1x}(x_1) + (a_{ijkl}(\mathbf{y})Y^{(\alpha)}_{k,ly}(\mathbf{y})$$
$$+ a_{ijk1}(\mathbf{y})X^{1\alpha}_k(\mathbf{y}))w^{(-1)}_{\alpha,1x}(x_1) + (a_{ij\beta\bar{\beta}}(\mathbf{y})s_\beta + a_{ijkl}(\mathbf{y})X^3_{k,ly}(\mathbf{y}))\varphi_{,1x}(x_1)$$
$$+(a_{ijkl}(\mathbf{y})Z_{k,ly}(\mathbf{y}) + a_{ijk1}(\mathbf{y})X^3_k(\mathbf{y}))\varphi^{(-1)}_{,1x1x}(x_1) + (a_{ijkl}(\mathbf{y})T^{(1)}_{k,ly}(\mathbf{y}) + a_{ijk1}(\mathbf{y})\mathcal{F}_k(\mathbf{y}))\theta_{,1x}(x_1). \tag{37}$$

Integrating (37) with respect to PC P_1, we obtain

$$N_{11}^{(-3)} = A^{00}_{11}U_{1,1x} + A^{01}_{11\alpha}u^{(0)}_{\alpha,1x1x} + D_{111}u^{(0)}_{1,1x1x} + D_{11\alpha}w^{(-1)}_{\alpha,1x} + I_{11}\varphi_{,1x} + G_{11}\varphi^{(-1)}_{,1x1x} + C_{11}\theta_{,1x}. \tag{38}$$

Multipling (37) by y_β, setting $j = 1$, and integrating the resulting relation for P_1, we find that

$$M_{i\beta}^{(-3)} = A^{10}_{i\beta}U_{1,1x} + A^{11}_{i\beta\alpha}u^{(0)}_{\alpha,1x1x} + d_{i\beta1}u^{(0)}_{1,1x1x} + d_{i\beta\alpha}w^{(-1)}_{\alpha,1x} + J_{i\beta}\varphi_{,1x} + g_{i\beta}\varphi^{(-1)}_{,1x1x} + c_{i\beta}\theta_{,1x}, \tag{39}$$

where

$$C_{ij} = \langle a_{ijkl}(\mathbf{y})T^{(1)}_{k,ly} + a_{ijk1}(\mathbf{y})\mathcal{F}_k(\mathbf{y})\rangle; \qquad D_{ijp} = \langle a_{ijkl}(\mathbf{y})Y^{(p)}_{k,ly}(\mathbf{y}) + a_{ijk1}(\mathbf{y})X^{1p}_k(\mathbf{y})\rangle;$$
$$G_{ij} = \langle a_{ijkl}(\mathbf{y})Z_{k,ly}(\mathbf{y}) + a_{ijk1}(y)X^3_k(\mathbf{y})\rangle; \qquad c_{i\beta} = \langle (a_{i1kl}(\mathbf{y})T^{(1)}_{k,ly}(\mathbf{y}) + a_{i1k1}(y)\mathcal{F}_k(\mathbf{y}))y_\beta\rangle;$$
$$d_{i\beta p} = \langle (a_{i1kl}(\mathbf{y})Y^{(p)}_{k,ly}(\mathbf{y}) + a_{i1k1}(\mathbf{y})X^{1p}_k(\mathbf{y}))y_\beta\rangle; \qquad g_{i\beta} = \langle (a_{i1kl}(\mathbf{y})Z_{k,ly}(\mathbf{y}) + a_{i1k1}(y)X^3_k(\mathbf{y}))y_\beta\rangle.$$

The solution of problem (25)–(28) allows us to compute the functions $u_1^{(0)}$, $w_\alpha^{(-1)}$ and $\varphi^{(-1)}$ from the asymptotic expansion (3). Thus, for displacements we get the expression $\mathbf{u}^\varepsilon \approx \varepsilon^{-1}w_\alpha^{(-1)}(x_1)\mathbf{e}_\alpha - y_\alpha w^{(-1)}_{\alpha,1x}(x_1)\mathbf{e}_1 + \varepsilon^{-1}y_{\bar{\beta}}s_\beta\mathbf{e}_\beta\varphi^{(-1)} + u_1^{(0)}\mathbf{e}_1$, where functions on the right-hand side are known (if the solution of the limiting problem is known). The formal exactness of the limiting problem (25)–(28) is equal to ε for displacements and ε^{-3} for stresses.

Cell problems can be solved either numerically or with the help of special methods developed, for example, in [15, 20–24].

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